

From algebraic curve to minimal surface and back

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ABSTRACT: We derive the Lax operator for a very large family of classical minimal surface solutions in AdS_3 describing Wilson loops in $\mathcal{N} = 4$ SYM theory. These solutions, constructed by Ishizeki, Kruczenski and Ziaman, are associated with a hyperelliptic surface of odd genus. We verify that the algebraic curve derived from the Lax operator is indeed none-other than this hyperelliptic surface.

KEYWORDS: Wilson, 't Hooft and Polyakov loops, AdS-CFT Correspondence, Integrable Field Theories

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1 Introduction

Integrability has led in recent years to great advances in the detailed understanding of the AdS/CFT correspondence [1] between free IIB superstring theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) in the planar approximation. The most well-studied sector is the spectrum of single-trace operators in the field theory and of the dual closed strings. Other advancements are in calculating scattering amplitudes, structure constants and open string states.

Open strings may end on D-branes or extend till the boundary of space, in which case they are dual to Wilson loop operators in the field theory. Studying them is more involved than solving the usual spectral problem for closed (or open) strings; at the classical level, the boundary conditions are not periodic (or Dirichlet/Neumann for fixed directions), but vary arbitrarily along the boundary. Solutions are not classified only by initial data, but by a rich structure of boundary conditions. If the Wilson loop is made of light-like segments, the regularized action on the surface (though not its shape) can be computed by solving a set of TBA-like equations [2]. In this paper we focus on a different case, when the Wilson

loop is space-like and piece together different integrability approaches that have been used to study them, verifying their consistency.

The algebraic curve approach [3] provides a framework to study classical closed strings by constructing the monodromy associated to a flat connection. This monodromy serves as a generating function of an infinite tower of conserved charges, including the energy. Solutions can be classified in terms of a spectral curve, which is naturally defined from the monodromy matrix.

The analog story for open strings dual to Wilson loops is far less well understood. There are no non-trivial cycles on the world-sheet, which makes it harder to construct a monodromy matrix. In fact it is often stated that it is impossible to construct one, when actually this can be done by either considering an open boundary-to-boundary monodromy, or by taking the monodromy including reflections from two boundary points. This latter approach was indeed implemented in [4] (based on [5]) for open strings ending on D-branes in $AdS_5 \times S^5$. Still, the analog construction for open strings ending on the boundary has not been found, so there is no satisfactory monodromy matrix for the strings dual to Wilson loops.

Complimentary approaches to the study of such surfaces were proposed in [6] and [7]. In the theory of integrable systems, the monodromy of an infinite-dimensional system plays an analogous role to that of the Lax operator in finite-dimensional systems. In particular one may extract an algebraic curve from the Lax operator, as the spectral curve is extracted from the monodromy. One may however also construct a Lax operator for infinite-dimensional systems when restricting oneself to a finite subsystem. This Lax operator is local, in contrast to a monodromy. As such, it is not unique and *à priori* neither is the resulting algebraic curve. Furthermore, the prescription for generating the conserved quantities from this operator is typically unclear. The constructions of [6, 7] give an algebraic curve based on a Lax operator for classical strings in AdS_3 . In this paper we extend their construction from genus-one solutions to higher genus.

We implement this on a very large class of explicit solutions in Euclidean AdS_3 which were constructed in [8] and [9] (see also [10], for a different approach to these solutions). These solutions are expressed in terms of theta functions associated with a hyperelliptic curve of arbitrary odd genus. Furthermore, they depend on a spectral parameter, such that each curve gives rise to a one-parameter family of classical solutions.¹ The purpose of our study is to verify that the algebraic curve associated with these surfaces is indeed the hyperelliptic curve they were constructed from and elucidate the role of the spectral parameter.

We adopt a modified version of the approach in [6] to construct the Lax operator for the full family of solutions in [8, 9] and derive the algebraic curve from it. Specifically, we construct a Lax operator whose Lax connection is the Pohlmeyer reduced connection rather than the sigma-model connection. Furthermore, we use the spectral parameter of [8], rather than the sigma-model spectral parameter used in [6].

¹There is a further discrete choice of possible boundaries for each curve and spectral parameter, see [8], but we shall not discuss it.

The structure of the paper is as follows: we begin in section 2 by introducing the coset manifold description of Euclidean AdS_3 . As the solutions of [8] are expressed in terms of Riemann theta functions, we devote section 3 to introducing these functions and some of their properties. In section 4, we construct the Lax pair and derive from it the algebraic curve. Finally, we demonstrate the construction for genus-one in section 5.

While most of the discussion is self-contained, we chose for brevity not to review all the details of the construction in [8, 9] and refer the reader there for where we have not. Our notation is mostly the same, with some exceptions which should not cause too much trouble for the reader.

2 Euclidean AdS_3 sigma model

We briefly review here the construction of the Euclidean AdS_3 (\mathbb{H}_3) sigma-model. The integrability of this subsector of the $AdS_5 \times S^5$ sigma model is most manifest when considered as the coset manifold $SL(2; \mathbb{C})/SU(2)$. It is parametrized by the group elements

$$g = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix} \in SL(2; \mathbb{C})/SU(2), \quad (2.1)$$

where X_i are the embedding coordinates of \mathbb{H}_3 in $\mathbb{R}^{1,3}$. The $SU(2)$ factor relates to the hermiticity of g . The sigma-model action is written in terms of $\mathfrak{sl}(2; \mathbb{C})/\mathfrak{su}(2)$ Maurer-Cartan forms as

$$S = \frac{1}{4\pi} \int d^2w \operatorname{tr}(\mathcal{J} \bar{\mathcal{J}}), \quad (2.2)$$

where w and \bar{w} are complex world-sheet coordinates (with ∂ and $\bar{\partial}$ their respective derivatives) and $\mathcal{J} = g^{-1} \partial g$ and $\bar{\mathcal{J}} = g^{-1} \bar{\partial} g$. The sigma-model equations of motion and Virasoro constraints can then be written in terms of these currents.

Instead of using these currents, we choose, following [8], to employ the Pohlmeyer reduction of this model [11]. Due to the hermiticity of g , we may perform the decomposition

$$g = h h^\dagger, \quad h \in SL(2; \mathbb{C}). \quad (2.3)$$

Two $\mathfrak{sl}(2; \mathbb{C})$ currents may naturally be associated with h via $j \equiv h^{-1} \partial h$ and $\bar{j} \equiv h^{-1} \bar{\partial} h$. As a consequence of their definition, the Virasoro constraints and the equations of motion, these currents satisfy

$$\operatorname{Tr} j = \operatorname{Tr} \bar{j} = 0, \quad \bar{\partial} j - \partial \bar{j} + [\bar{j}, j] = 0, \quad (2.4a)$$

$$\det(\bar{j} + j^\dagger) = 0, \quad (2.4b)$$

$$\partial(\bar{j} + j^\dagger) + \frac{1}{2} [j - \bar{j}^\dagger, \bar{j} + j^\dagger] = 0. \quad (2.4c)$$

A *spectral parameter*, $x \in \mathbb{C}$ may be introduced, as in [6, 7], to define new currents

$$J(x) = \frac{1}{1-x} (j + x \bar{j}^\dagger), \quad \bar{J}(x) = \frac{1}{1+x} (\bar{j} - x j^\dagger), \quad (2.5)$$

such that the equations for j and \bar{j} , (2.4a) and (2.4c), are equivalent to the flatness condition for J and \bar{J}

$$\bar{\partial}J - \partial\bar{J} + [\bar{J}, J] = 0. \quad (2.6)$$

While it is clear that (2.4a) is satisfied by $J(x)$ and $\bar{J}(x)$ for all x , a key fact is that for $\text{Re}(x) = 0$ they also solve the equations of motion, (2.4c). Thus for different values of imaginary x the currents $J(x)$ and $\bar{J}(x)$ represent a family of real physical solutions of the sigma-model and the spectral parameter is not merely a formal expansion parameter.

An explicit parameterization of the currents j and \bar{j} was given in [8]

$$j = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha} \\ e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \quad \bar{j} = \begin{pmatrix} \frac{1}{2}\bar{\partial}\alpha & e^{\alpha} \\ -e^{-\alpha} & -\frac{1}{2}\bar{\partial}\alpha \end{pmatrix}, \quad (2.7)$$

where $\alpha(w, \bar{w})$ is a real-valued function of the world-sheet coordinates, satisfying the cosh-Gordon equation

$$\partial\bar{\partial}\alpha = 2 \cosh(2\alpha). \quad (2.8)$$

Substituting (2.7) into (2.5), we find the currents

$$J = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha} \\ \frac{1+x}{1-x}e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \quad \bar{J} = \begin{pmatrix} \frac{1}{2}\bar{\partial}\alpha & \frac{1-x}{1+x}e^{\alpha} \\ -e^{-\alpha} & -\frac{1}{2}\bar{\partial}\alpha \end{pmatrix}. \quad (2.9)$$

Indeed it was already noticed in [8] that for $\lambda = \frac{1+x}{1-x}$ on the unit circle, this describes real solutions, which exactly corresponds to imaginary x .² x serves thus in a dual role: shifts along the imaginary axis lead to different solutions, while expanding the flatness condition around any such point leads to the usual equations of motion.

The cosh-Gordon equation can be solved in terms of theta functions of hyperelliptic Riemann surfaces. Given such a solution along with a choice of x we have the currents J and \bar{J} and can reconstruct the full solution by reversing the procedure above. We first identify the flatness condition in (2.6) as the compatibility condition for the auxiliary linear problem

$$\partial\psi = \psi J, \quad \bar{\partial}\psi = \psi \bar{J}. \quad (2.10)$$

For imaginary x this connection also satisfies the equations of motion (2.4c) so we may then identify $h(x)$ in (2.3) with the matrix made of the two linearly independent solutions to the auxiliary problem. From $h(x)$, we can then construct the group element $g(x)$ (2.3), giving the solution to the \mathbb{H}_3 sigma-model. Thus, changing the spectral parameter enables us to construct from J and \bar{J} a solution given by the group element $g(x)$ *at any point along the imaginary-axis*.

3 Riemann theta functions

The procedure outlined above gives solutions to the \mathbb{H}_3 sigma-model from solutions of the cosh-Gordon equation. The fact that this equation can be solved by theta functions

²The same form of spectral parameter was found previously. See e.g., [12].

associated with hyperelliptic Riemann surfaces of odd genus [13] allows to find a very large family of solutions to the sigma-model [8]. We provide a quick review to theta functions, focusing on properties relevant for our purposes. A lot more can be found in the abundant literature on the subject, e.g., [14–18].

The surface is a hyperelliptic curve defined by a function $f(\lambda)$ of the form³

$$f(\lambda) = \sqrt{\lambda} \prod_{i=1}^{2g} \sqrt{\lambda - \lambda_i}, \quad (3.1)$$

where $\lambda_i \in \mathbb{C}$, together with $\lambda = 0$ and $\lambda = \infty$, are the branch points of f . It is apparent that such surfaces are two-sheeted, corresponding to the two branches of the square root. Furthermore, to ensure real solutions to the sigma-model, we require the surface to be equipped with an *anti-holomorphic involution*, $\tau : \lambda \mapsto -1/\bar{\lambda}$, and in particular the set of $2g + 2$ branch points $\{\lambda_i\}$ is invariant under this involution.

For any Riemann surface Σ there exists a canonical basis of homology cycles $\{a_i, b_i\}$ with $i = 1, \dots, g$, satisfying $a_i \circ a_j = b_i \circ b_j = 0$, and, $a_i \circ b_j = \delta_{ij}$. A set of cohomology one-forms $\{\omega_i\}$ dual to $\{a_i\}$, i.e., satisfying $\oint_{a_i} \omega_j = \delta_{ij}$, may be defined. With this one constructs the *period matrix*

$$\Pi_{ij} = \oint_{b_i} \omega_j. \quad (3.2)$$

This is a symmetric $g \times g$ matrix with positive definite imaginary part.

Riemann theta functions are defined as

$$\theta(\Pi; z) \equiv \sum_{n \in \mathbb{Z}^g} \exp \left(2\pi i \left(\frac{1}{2} n^T \Pi n + n^T z \right) \right), \quad (3.3)$$

where $z \in \mathbb{C}^g$ and Π is a $g \times g$ symmetric matrix with positive definite imaginary part. We consider the Riemann theta function associated with Σ by taking Π to be the period matrix (3.2). We keep throughout a fixed Riemann surface and therefore write $\theta(z) \equiv \theta(\Pi; z)$.

The $g \times 2g$ matrix (I, Π) (where I is the $g \times g$ identity matrix) generates a lattice, denoted by $\mathcal{L}(\Sigma) = (I, \Pi) \times \mathbb{Z}^{2g}$. The *Jacobian variety* of the surface Σ is the quotient $\mathfrak{J}(\Sigma) = \mathbb{C}^g / \mathcal{L}(\Sigma)$. One may then define the *Abel-Jacobi mapping*

$$\phi : \Sigma \rightarrow \mathfrak{J}(\Sigma), \quad \phi(\lambda) = \int_{\lambda_0}^{\lambda} \omega, \quad (3.4)$$

where λ_0 is the *base point* of the mapping, which we choose to be at $\lambda_0 = 0$. We often represent the Abel-Jacobi mapping by the integral $\int_0^{\lambda} \omega \equiv \phi(\lambda)$.

Strictly, we should distinguish between points $p \in \Sigma$ and their projection λ_p onto the complex plane. We may represent the uplift to the Riemann surface as $p = (\lambda, f(\lambda))$, and denote the other uplift $p' = (\lambda, -f(\lambda))$ (at branch points they are, of course, degenerate). The endpoint of integration in (3.4) is really not λ itself, but one of its uplifts. We will often use this notation and label corresponding points on the two sheets as λ and λ' .

³We will also refer to the function itself, and not only its solutions, as the curve.

It is useful to define theta functions with *characteristics*. Given $n, m \in \mathbb{C}^g$, the theta function with these characteristics is given by

$$\theta \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] (z) = e^{\pi i (\frac{1}{4} n^\top \Pi n + n^\top z + \frac{1}{2} n^\top m)} \theta \left(z + \frac{1}{2} m + \frac{1}{2} \Pi n \right). \quad (3.5)$$

We will concern ourselves only with integer characteristics $n, m \in \mathbb{Z}^g$. From (3.5) and (for the second equality) the definition (3.3) we find

$$\theta \left[\begin{smallmatrix} 2n \\ 2m \end{smallmatrix} \right] (z) = e^{\pi i (n^\top \Pi n + 2n^\top z + 2n^\top m)} \theta(z + m + \Pi n) = \theta(z). \quad (3.6)$$

The theta functions are therefore quasi-periodic so they are defined, up to monodromies, on the Jacobian. An integer characteristic is said to be *odd* or *even* if the scalar product, $n^\top m$, is odd or even. We note that theta functions with odd characteristics are odd functions of $z \in \mathbb{C}^g$ and theta functions with even characteristics are even.

Let us consider now the Abel-Jacobi map more explicitly. The differentials in (3.2) are given, up to a normalization, by [18]

$$\omega_i(\lambda) = \frac{\lambda^{i-1} d\lambda}{f(\lambda)}, \quad (3.7)$$

with $f(\lambda)$ in (3.1). Since $f(\lambda') = -f(\lambda)$, then also $\omega(\lambda') = -\omega(\lambda)$ and consequently, $\int_0^{\lambda'} = -\int_0^\lambda$.

Despite the fact that the Abel-Jacobi mapping is not single-valued (it depends on the contour of integration), the composite map $\theta \circ \phi : \Sigma \rightarrow \mathbb{C}$ has a well-defined set of zeros. In particular, it has g zeros and no poles [18]. The same is clearly true with characteristics. There are $2g + 2$ branch points of the algebraic curve defining the Riemann surface. To each of these branch points we may naturally associate a characteristic. The integral $\int_0^{\lambda_i}$ is half of the period integral between 0 to λ_i and back to 0 on the second sheet. Therefore it can be expressed as $\int_0^{\lambda_i} = \frac{1}{2} m I + \frac{1}{2} \Pi n$, for some $m, n \in \mathbb{Z}$. Now, applying (3.5)

$$\theta \left(\int_0^{\lambda_i} \right) = e^{\pi i (-\frac{1}{4} n^\top \Pi n - \frac{1}{2} n^\top m)} \theta \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] (0), \quad (3.8)$$

If $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ is an odd characteristic, then $\theta \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] (z)$ is an odd function and thus $\theta(\int_0^{\lambda_i})$ vanishes. We refer to branch points for which the corresponding characteristics are odd, as *odd branch points* (and correspondingly for even characteristics). We denote by λ_i^- the odd branch points and by λ_i^+ the even branch points. There are precisely g odd branch points of a hyperelliptic surface, corresponding to the g zeros of $\theta \circ \phi$ [18]. We fixed our curve to have branch points at 0 and at ∞ and a special role will be played by the theta functions with the appropriate characteristics. For 0, this is the original theta function. The other one is defined as $\hat{\theta}(z) = \theta \left[\begin{smallmatrix} k \\ l \end{smallmatrix} \right] (z)$, where $2 \int_0^\infty = l + \Pi k$.

$2 \int_0^{\lambda_i} = I m + \Pi n$ with $m, n \in \mathbb{Z}$ is a full period. Combining this with the fact that $\hat{\theta}(0) = 0$ implies $\hat{\theta}(2 \int_0^\lambda) = 0$. In fact for any integer p , a theta functions of the form $\theta \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] (p \int_0^\lambda)$, has $p^2 g$ zeros (including multiplicities) [18]. As we show in appendix B, in the case of $\hat{\theta}(2 \int_0^\lambda)$ these are indeed just the $2g + 2$ branch points, of which $g - 1$ (the finite odd

ones) are triple zeros and the other $g + 3$ branch points are simple zeros on the Riemann surface.

Let us mention one more necessary tool: *Directional derivatives* with respect to a point λ on the Riemann surface are defined as

$$D_\lambda \theta(z) \equiv \omega(\lambda) \cdot \frac{\partial}{\partial z} \theta(z). \quad (3.9)$$

4 The Lax operator and algebraic curve

Recall that a Lax operator is part of a Lax pair (L, M) , where the evolution of the system is described by

$$\frac{dL}{dt} = [L, M], \quad (4.1)$$

with t some ‘time’ parameter. For infinite-dimensional systems, such as the sigma-model, we may treat the two world-sheet coordinates as independent ‘time’ parameters and study

$$dL = [L, M], \quad (4.2)$$

where d is the exterior derivative with respect to the world-sheet coordinates of the open string, $d = dw \partial + d\bar{w} \bar{\partial}$. An algebraic curve is naturally associated to this Lax operator via the characteristic polynomial

$$\det(L - yI) = 0. \quad (4.3)$$

For the \mathbb{H}_3 sector this simplifies to

$$y^2 = -\det L. \quad (4.4)$$

The Lax equations ensure that this curve is independent of the world-sheet position.

We now proceed to employ the method proposed in [6] (a variation of a theorem in [19]) to find the Lax operator and from it the algebraic curve for the minimal surface solutions discussed above. It is natural that the result should be the algebraic curve (3.1) associated to the theta function, but it is rather opaque how this would come about. The prescription in [6] is to take the Lax operator as

$$L(w, \bar{w}; x) = \hat{\Psi}(w, \bar{w}; x)^{-1} \cdot A(x) \cdot \hat{\Psi}(w, \bar{w}; x), \quad (4.5)$$

where $\hat{\Psi}(w, \bar{w}; x)$ is the matrix whose rows are the linearly independent solutions of an auxiliary linear problem of the form,

$$\partial \psi = \psi \mathcal{J}(x), \quad \bar{\partial} \psi = \psi \bar{\mathcal{J}}(x). \quad (4.6)$$

and the matrix $A(x)$ is determined by the requirement that $L(w, \bar{w}; x)$ have polynomial dependence on the spectral parameter. Here

$$\mathcal{J}(x) = \frac{\mathcal{J}}{1-x}, \quad \bar{\mathcal{J}}(x) = \frac{\bar{\mathcal{J}}}{1+x}, \quad (4.7)$$

are defined in terms of the Maurer-Cartan forms \mathcal{J} and $\bar{\mathcal{J}}$ in (2.2).

We wish to modify this prescription for our purposes. Firstly, we will construct a Lax operator with respect to the Pohlmeyer reduced connection j and \bar{j} , i.e., for $M = J(\lambda)dw + \bar{J}(\lambda)d\bar{w}$, rather than $M = \mathcal{J}(x)dw + \bar{\mathcal{J}}(x)d\bar{w}$. It is then natural to replace x in the above argument by $\lambda = \frac{1+x}{1-x}$. While an algebraic curve in terms of x is clearly related by a birational transformation to one in terms of λ (and therefore structurally equivalent), the condition for polynomiality of the Lax matrix in λ or in x are inequivalent. The sigma-model connection and the Pohlmeyer reduced connection are related by a gauge transformation [12], and we thus expect the resulting spectral curve to be the same. We demonstrate the equivalence of solving the respective Lax equations in appendix C.

We shall proceed to construct the Lax operator and the corresponding algebraic curve. Beforehand, however, we need a further result from [8]; the solutions to the auxiliary linear problem. Recall that $h(w, \bar{w}; \lambda)$ is made of the two independent solutions and is given by [8]

$$h(w, \bar{w}; \lambda) = \left(\frac{\theta^2(\int_0^\lambda)}{2\hat{\theta}(2\int_0^\lambda)D_\infty\hat{\theta}(0)} \right)^{1/2} \times \begin{pmatrix} \sqrt{-\lambda} \frac{\hat{\theta}(z+\int_0^\lambda)}{\hat{\theta}(z)} \left(\frac{\hat{\theta}(z)}{\hat{\theta}(z)} \right)^{1/2} e^{\mu(\lambda)w+\nu(\lambda)\bar{w}} & \frac{\theta(z+\int_0^\lambda)}{\theta(z)} \left(\frac{\theta(z)}{\hat{\theta}(z)} \right)^{1/2} e^{\mu(\lambda)w+\nu(\lambda)\bar{w}} \\ -\sqrt{-\lambda} \frac{\hat{\theta}(z-\int_0^\lambda)}{\hat{\theta}(z)} \left(\frac{\hat{\theta}(z)}{\hat{\theta}(z)} \right)^{1/2} e^{-\mu(\lambda)w-\nu(\lambda)\bar{w}} & \frac{\theta(z-\int_0^\lambda)}{\theta(z)} \left(\frac{\theta(z)}{\hat{\theta}(z)} \right)^{1/2} e^{-\mu(\lambda)w-\nu(\lambda)\bar{w}} \end{pmatrix}, \quad (4.8)$$

where $z = z(w, \bar{w})$ is a function of the world-sheet coordinates. The precise form of $z(w, \bar{w})$, of $\mu(\lambda)$ and of $\nu(\lambda)$ will not be important for our discussion. As we expect the curve to be hyperelliptic, we make the ansatz [19]

$$A(\lambda) = g(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.9)$$

The resulting Lax operator is given by

$$L(w, \bar{w}; \lambda) \equiv \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad (4.10)$$

where

$$L_{11} = -L_{22} = \frac{\sqrt{-\lambda}\theta^2(\int_0^\lambda)g(\lambda)}{2\hat{\theta}(2\int_0^\lambda)D_\infty\hat{\theta}(0)} \left(\frac{\hat{\theta}(z+\int_0^\lambda)\theta(z-\int_0^\lambda) - \theta(z+\int_0^\lambda)\hat{\theta}(z-\int_0^\lambda)}{\theta(z)\hat{\theta}(z)} \right), \quad (4.11a)$$

$$L_{12} = \frac{\theta^2(\int_0^\lambda)g(\lambda)}{\hat{\theta}(2\int_0^\lambda)D_\infty\hat{\theta}(0)} \frac{\theta(z+\int_0^\lambda)\theta(z-\int_0^\lambda)}{\theta(z)\hat{\theta}(z)}, \quad (4.11b)$$

$$L_{21} = -\frac{\lambda\theta^2(\int_0^\lambda)g(\lambda)}{\hat{\theta}(2\int_0^\lambda)D_\infty\hat{\theta}(0)} \frac{\hat{\theta}(z+\int_0^\lambda)\hat{\theta}(z-\int_0^\lambda)}{\theta(z)\hat{\theta}(z)}. \quad (4.11c)$$

We now emphasize a point about this construction. It is apparent that $g(\lambda)$ must be the algebraic curve as, by the definition of L , $\det L = -g(\lambda)^2$. This observation advocates the converse procedure of that in [6], i.e., we make an ansatz for $g(\lambda)$, and we need to verify the polynomiality of the resulting Lax matrix. In order to do that we need algebraic relations between $f(\lambda)$ in (3.1) and the theta functions.

4.1 A meromorphic function

Consider the function on the Riemann surface Σ , given by

$$G(p) = \frac{\hat{\theta}(2 \int_0^{\lambda_p})}{\theta^4(\int_0^{\lambda_p})}. \quad (4.12)$$

Recall that $p = (f(\lambda_p), \lambda_p)$, where λ_p is the projection of p onto the complex plane. First note that it is single valued on Σ and in particular, it is independent of the contour of integration. To see that, consider the monodromy around a closed cycle $\int_0^{\lambda_p} \rightarrow \int_0^{\lambda_p} + m + \Pi n$, for $m, n \in \mathbb{Z}$. Applying (3.6) and (3.5) we find

$$G(p) \rightarrow \frac{\hat{\theta}(2 \int_0^{\lambda_p} + 2m + 2\Pi n)}{\theta^4(\int_0^{\lambda_p} + m + \Pi n)} = \frac{\hat{\theta}(2 \int_0^{\lambda_p})}{\theta^4(\int_0^{\lambda_p})}, \quad (4.13)$$

Since both θ and $\hat{\theta}$ are analytic, $G(p)$ is meromorphic on Σ .

If instead we consider G as a function on the complex λ -plane, it will have the same branch-cuts as $f(\lambda)$, and then $G^2(\lambda)$ is meromorphic on the complex λ -plane.

One may be tempted to guess that $G(\lambda) = f(\lambda)$, but this is not the case. The zeros of the numerator and denominator have already been discussed in section 3 and appendix B. Combining these results, we see that G^2 has⁴

- simple zeros at the $g + 2$ even branch points, $\{\lambda_i^+\}$,
- simple poles at the $g - 1$ odd finite branch points, $\{\lambda_i^-\}$,
- a third order pole at ∞ ,
- no other zeros or poles.

We may therefore identify G as

$$G(\lambda) = G_0 \frac{\prod_i^{g+2} \sqrt{\lambda - \lambda_i^+}}{\prod_i^{g-1} \sqrt{\lambda - \lambda_i^-}}. \quad (4.14)$$

We can determine the constant, though it is not really crucial for our purposes, from the asymptotics of the theta functions

$$\lim_{\lambda \rightarrow \infty} \lambda^{-3/2} G(\lambda) = \frac{1}{4} \left(\frac{1}{D_\infty \hat{\theta}(0)} \right)^3. \quad (4.15)$$

Recall that [8]

$$\sqrt{-\lambda} = 2 \frac{D_\infty \hat{\theta}(0) \hat{\theta}(\int_0^\lambda)}{\theta(0) \theta(\int_0^\lambda)}, \quad (4.16)$$

⁴Since all the zero and poles are branch points of $G(\lambda)$, the same statements actually hold for G as a function on Σ .

So we find that

$$G_0 = \frac{D_0 \hat{\theta}(0)}{\left(\theta(0) D_\infty \hat{\theta}(0)\right)^2}. \quad (4.17)$$

Splitting the product (3.1) into terms with the odd finite branch points $\{\lambda_i^-\}$ and with the even branch points $\{\lambda_i^+\}$ we find

$$g(\lambda) = \frac{G(\lambda)}{G_0} \prod_i^{g-1} (\lambda - \lambda_i^-) = \sqrt{\lambda} \prod_i^{g-1} \sqrt{\lambda - \lambda_i^-} \prod_j^{g+1} \sqrt{\lambda - \lambda_j^+} = f(\lambda). \quad (4.18)$$

So while $G(\lambda)$ is not equal to $f(\lambda)$, their ratio is a polynomial.

We now use this representation of $f(\lambda)$ to prove that the entries of the Lax matrix (4.11) are indeed polynomials in λ and find the order of these polynomials.

4.2 L_{11}

Using $\int_0^{\lambda'} = -\int_0^\lambda$ where the integrals are along image paths on the two sheets, we can rewrite (4.11a) as

$$L_{11} = \frac{\theta^2(0) D_\infty \hat{\theta}(0)}{2 D_0 \hat{\theta}(0)} \frac{\hat{\theta}(z + \int_0^\lambda) \theta(z - \int_0^\lambda) - \hat{\theta}(z + \int_0^{\lambda'}) \theta(z - \int_0^{\lambda'})}{\theta(z) \hat{\theta}(z)} \sqrt{-\lambda} \frac{\prod_i^{g-1} (\lambda - \lambda_i^-)}{\theta^2(\int_0^\lambda)}, \quad (4.19)$$

- **No branch cuts:** clearly the $\sqrt{-\lambda}$ term introduces a branch cut which changes sign between the two sheets $\lambda \rightarrow \lambda'$, but this is negated by the branch cut of the second fraction (see appendix A). Therefore overall L_{11} is single valued.
- **Analyticity:** as discussed in section 3, the last denominator has zeros at the odd finite branch points and at infinity. These are simple zeros which are canceled by corresponding zeros of the polynomial in the numerator.
- **Order:** near the origin the two terms with branch cuts in (4.19) mean that $L_{11}(0) = 0$, so L_{11} has no constant piece. At large λ , it is shown in appendix A that the second fraction scales together like $\lambda^{-1/2}$, as does $\theta(\int_0^\lambda)$. We find that $L_{11} = O(\lambda^g)$ and is therefore a degree g polynomial in λ .

4.3 L_{12}

$$L_{12} = \frac{\theta^2(0) D_\infty \hat{\theta}(0)}{D_0 \hat{\theta}(0)} \frac{\theta(z + \int_0^\lambda) \theta(z - \int_0^\lambda)}{\theta(z) \hat{\theta}(z)} \frac{\prod_i^{g-1} (\lambda - \lambda_i^-)}{\theta^2(\int_0^\lambda)}, \quad (4.20)$$

- **No branch cuts:** L_{12} is invariant under $\lambda \rightarrow \lambda'$.
- **Analyticity:** the only possible zeros of the denominator are at λ_i^- and are cancelled by those in the numerator.
- **Order:** the z -dependant fraction has a finite limit at large λ . The last one scales like λ^g , so L_{12} is a polynomial of degree g .

4.4 L_{21}

$$L_{21} = -\frac{\theta^2(0)D_\infty\hat{\theta}(0)}{D_0\hat{\theta}(0)}\frac{\hat{\theta}(z+\int_0^\lambda)\hat{\theta}(z-\int_0^\lambda)}{\theta(z)\hat{\theta}(z)}\frac{\lambda\prod_i^{g-1}(\lambda-\lambda_i^-)}{\theta^2(\int_0^\lambda)}, \quad (4.21)$$

- The same arguments as in the case of L_{12} show that L_{21} is a polynomial of degree $g+1$ with no constant term.

We conclude that with the representation (4.18) of $f(\lambda)$, the Lax operator $L(w, \bar{w}, \lambda)$ given by (4.11) is indeed the Lax operator describing the family of solutions and its determinant gives the corresponding algebraic curve. This implies that the procedure advocated in [6] indeed applies to the open string solutions of [8] with the spectral parameter $\lambda = (1+x)/(1-x)$ and as expected, the resulting curve is none other than the hyperelliptic curve defining the Riemann theta functions.

5 The genus-one case

We have constructed the Lax operator for a solution based on an arbitrary genus curve, shown that its entries are polynomial in λ and proven that the resulting algebraic curve is indeed $f(\lambda)$ (3.1). We have not presented, though, explicit expressions for the coefficients of the polynomials in λ in the different matrix elements of L . To do that would require to disentangle expressions like $\theta(z+\int_0^\lambda)\theta(z-\int_0^\lambda)$. In fact the addition theorem (see e.g., [17]) may be employed to do just that. This allows one to write such products as sums over products of theta functions of z and theta functions of λ , thus splitting the spectral parameter and world-sheet dependence in (4.11). It does so at the expense of introducing a sum over all possible integer (mod 2) characteristics. We have not found the resulting expressions for arbitrary genus illuminating, but in the case of genus-one, which we present here, they are very explicit and rather simple

For genus-one the Riemann theta functions reduce to elliptic theta functions, see [9]

$$\theta(z) = \vartheta_3(\pi z; q), \quad \hat{\theta}(z) = -\vartheta_1(\pi z; q), \quad \text{with } q = e^{i\pi\Pi}. \quad (5.1)$$

The period matrix is given by $\Pi = i\mathbb{K}(k')/\mathbb{K}(k)$, where \mathbb{K} is the complete elliptic integral of the first kind, and k and k' are the elliptic modulus and complementary elliptic modulus, respectively.

Employing the result for $f(\lambda)$ from section 4, we have

$$f(\lambda) = \sqrt{\lambda}\sqrt{\lambda-\lambda_1}\sqrt{\lambda-\lambda_2} = -\vartheta_3^2(0)D_\infty\vartheta_1(0)\frac{\vartheta_1(2\int_0^\lambda)}{\vartheta_3^4(\int_0^\lambda)}, \quad (5.2)$$

where $\lambda_2 = -1/\bar{\lambda}_1$. We have absorbed a factor of π in the definition of the elliptic functions and used that for genus-one $\omega(0) = -\omega(\infty)$. Note that the cross-ratio of $\{0, \infty, \lambda_1, -1/\bar{\lambda}_1\}$ is always real, so they are all along a line, which without loss of generality we take to be the real line. Furthermore the only odd branch point is at infinity. We also note that the elliptic modulus and complementary modulus may be expressed in terms of the branch points via

$$k = \frac{a}{\sqrt{1+a^2}}, \quad k' = \frac{1}{\sqrt{1+a^2}}. \quad (5.3)$$

The expression for the Lax operator, (4.11), becomes

$$\begin{aligned}
 L_{11} &= -\frac{\sqrt{-\lambda}}{2} \vartheta_3^2(0) \frac{\vartheta_1(z + \int_0^\lambda) \vartheta_3(z - \int_0^\lambda) - \vartheta_3(z + \int_0^\lambda) \vartheta_1(z - \int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)}, \\
 L_{12} &= \vartheta_3^2(0) \frac{\vartheta_3(z + \int_0^\lambda) \vartheta_3(z - \int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)}, \\
 L_{21} &= -\lambda \vartheta_3^2(0) \frac{\vartheta_1(z + \int_0^\lambda) \vartheta_1(z - \int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)}.
 \end{aligned} \tag{5.4}$$

We would like to evaluate the coefficients of the polynomials, and to explicitly recover the spectral curve. For genus-one, the expression for the spectral parameter (4.16) simplifies to

$$\lambda = \frac{\vartheta_1^2(\int_0^\lambda)}{\vartheta_3^2(\int_0^\lambda)}. \tag{5.5}$$

The addition theorem at genus-one is particularly simple and may be employed to split the world-sheet and spectral parameter dependence

$$\begin{aligned}
 \vartheta_1\left(z + \int_0^\lambda\right) \vartheta_3\left(z - \int_0^\lambda\right) - \vartheta_3\left(z + \int_0^\lambda\right) \vartheta_1\left(z - \int_0^\lambda\right) &= 2 \frac{\vartheta_2(z) \vartheta_4(z) \vartheta_1(\int_0^\lambda) \vartheta_3(\int_0^\lambda)}{\vartheta_2(0) \vartheta_4(0)}, \\
 \vartheta_3\left(z + \int_0^\lambda\right) \vartheta_3\left(z - \int_0^\lambda\right) &= \frac{\vartheta_1^2(z) \vartheta_1^2(\int_0^\lambda) + \vartheta_3^2(z) \vartheta_3^2(\int_0^\lambda)}{\vartheta_3^2(0)}, \\
 \vartheta_1\left(z + \int_0^\lambda\right) \vartheta_1\left(z - \int_0^\lambda\right) &= \frac{\vartheta_1^2(z) \vartheta_3^2(\int_0^\lambda) - \vartheta_3^2(z) \vartheta_1^2(\int_0^\lambda)}{\vartheta_3^2(0)}.
 \end{aligned} \tag{5.6}$$

This gives

$$\begin{aligned}
 L_{11} &= -\sqrt{-\lambda} \frac{\vartheta_3^2(0)}{\vartheta_2(0) \vartheta_4(0)} \frac{\vartheta_2(z) \vartheta_4(z) \vartheta_1(\int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3(\int_0^\lambda)} = \mp i \frac{\lambda}{\sqrt{k k'}} \frac{\vartheta_2(z) \vartheta_4(z)}{\vartheta_3(z) \vartheta_1(z)}, \\
 L_{12} &= \frac{\vartheta_1^2(z) \vartheta_1^2(\int_0^\lambda) + \vartheta_3^2(z) \vartheta_3^2(\int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)} = \lambda \frac{\vartheta_1(z)}{\vartheta_3(z)} + \frac{\vartheta_3(z)}{\vartheta_1(z)}, \\
 L_{21} &= -\lambda \frac{\vartheta_1^2(z) \vartheta_3^2(\int_0^\lambda) - \vartheta_3^2(z) \vartheta_1^2(\int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)} = \lambda^2 \frac{\vartheta_3(z)}{\vartheta_1(z)} - \lambda \frac{\vartheta_1(z)}{\vartheta_3(z)},
 \end{aligned} \tag{5.7}$$

where we have used the identity $\vartheta_2(0) \vartheta_4(0) = \sqrt{k k'} \vartheta_3^2(0)$. We note that all the components of the Lax operator have polynomial dependence on the spectral parameter and with the orders of the polynomials matching the results of section 4.

Recall that, due to the tracelessness of L , we have that

$$\det(yI - L(w, \bar{w}; \lambda)) = 0 \quad \Leftrightarrow \quad y^2 = -\det L(w, \bar{w}; \lambda), \tag{5.8}$$

which is the equation of the algebraic curve. It is straightforward to read off the algebraic curve from (5.7)

$$y^2 = \lambda^3 + \lambda^2 \left\{ -\frac{1}{k k'} \left(\frac{\vartheta_2(z) \vartheta_4(z)}{\vartheta_3(z) \vartheta_1(z)} \right)^2 + \left(\frac{\vartheta_3(z)}{\vartheta_1(z)} \right)^2 - \left(\frac{\vartheta_1(z)}{\vartheta_3(z)} \right)^2 \right\} - \lambda. \tag{5.9}$$

We may express the elliptic theta functions of the second and fourth kinds here in terms of the first and third kinds via the identity

$$\vartheta_2^2(z)\vartheta_4^2(z) = kk' \left(\vartheta_3^4(z) - \vartheta_1^4(z) + \vartheta_3^2(z)\vartheta_1^2(z) \left[\frac{\vartheta_2^2(0)}{\vartheta_4^2(0)} - \frac{\vartheta_4^2(0)}{\vartheta_2^2(0)} \right] \right), \quad (5.10)$$

which is derived from the addition theorem. From this we find the nice expression for the curve

$$y^2 = \lambda \left(\lambda^2 + \lambda \left[\frac{\vartheta_4^2(0)}{\vartheta_2^2(0)} - \frac{\vartheta_2^2(0)}{\vartheta_4^2(0)} \right] - 1 \right). \quad (5.11)$$

A final identity giving the position of the cuts of the Riemann surface associated to the theta functions

$$\left(\frac{\vartheta_2(0)}{\vartheta_4(0)} \right)^2 = \lambda_1, \quad (5.12)$$

leads to

$$y^2 = \lambda (\lambda - \lambda_1) (\lambda + 1/\lambda_1). \quad (5.13)$$

We have thus recovered the genus-one algebraic curve explicitly from the Lax operator. We recover this curve via an alternative approach in appendix D. In [6, 7, 20] several algebraic curves of genus-zero or genus-one in \mathbb{H}_3 were constructed. It is straightforward to see that the curve (5.13) agrees with the curves found in these references.

6 Discussion

We have constructed a Lax operator and the algebraic curve for the most general minimal surface solution of [8], which holographically describe Wilson loop operators in \mathbb{R}^2 . This was done via a modification of the prescription of [6]. We found that the curve is given by the the same hyperelliptic curve defining the Riemann surface.

Though there are ambiguities in defining the Lax operator and the fact that it is local, it is still natural to expect the resulting curve to be unique, and not dependent on these ambiguities. This suggest that this curve plays the same role as that normally derived from the monodromy matrix.

It would of course be interesting to rederive these results from a monodromy matrix on the open string surface (including appropriate reflections from the boundaries). Monodromies have been constructed for open strings with certain integrable D-brane boundary conditions [4, 5], but no such construction has been successfully applied to strings ending on the boundary of AdS.

An important difference between the closed and open string cases is that given an algebraic curve there are many minimal surfaces associated to it, not related to each-other by a global symmetry. This is evident in the construction of [8], where the solution is given in terms of a curve, a continuous spectral parameter on the unit circle and a discrete choice among different possible boundaries for the string. We have seen it from the opposite point of view, where we indeed found the same algebraic curve for all these surfaces, irrespective of the value of the spectral parameter, which is just the phase (or imaginary part) of our full complex spectral parameter.

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A Taylor expansion of theta functions

For our analysis in section 4 we need to understand the behaviour of theta functions of the form $\theta(z \pm \int_0^\lambda)$ with λ near a branch point λ_i . As can be seen from (3.7) (see also [9]), the differentials are best expressed in terms of the variables $y = -i\sqrt{\lambda - \lambda_i}$, and then have a finite limit at $\lambda = \lambda_i$. It is straightforward to see that the integral \int_0^λ is an odd function of y . The Taylor expansion about $y = 0$ gives

$$\theta\left(z \pm \int_0^\lambda\right) = \theta\left(z \pm \int_0^{\lambda_i}\right) \pm y D_{\lambda_i} \theta\left(z \pm \int_0^{\lambda_i}\right) + \mathcal{O}(y^2). \quad (\text{A.1})$$

We are particularly interested in the following expansion about $\lambda = 0$

$$\hat{\theta}\left(z + \int_0^\lambda\right) \theta\left(z - \int_0^\lambda\right) - \theta\left(z + \int_0^\lambda\right) \hat{\theta}\left(z - \int_0^\lambda\right) = 2y(\theta(z) D_0 \hat{\theta}(z) - \hat{\theta}(z) D_0 \theta(z)) + \mathcal{O}(y^3), \quad (\text{A.2})$$

which we see behaves as $\sqrt{\lambda}$.

Similarly, for large λ we may expand about $w = -i/\sqrt{\lambda}$ and it is straightforward to see that

$$\hat{\theta}\left(z + \int_0^\lambda\right) \theta\left(z - \int_0^\lambda\right) - \theta\left(z + \int_0^\lambda\right) \hat{\theta}\left(z - \int_0^\lambda\right) = 2w(\theta(z) D_\infty \hat{\theta}(z) - \hat{\theta}(z) D_\infty \theta(z)) + \mathcal{O}(w^3), \quad (\text{A.3})$$

B The zeros of $\hat{\theta}(2 \int_0^\lambda)$

We now show that $\hat{\theta}(2 \int_0^\lambda)$ has triple zeros at the *odd finite branch points*, $\{\lambda_i^-\}$. As in the previous appendix, we may Taylor expand $\hat{\theta}(2 \int_0^\lambda)$ about a branch point λ_i (i.e., $y = 0$)

$$\hat{\theta}\left(2 \int_0^\lambda\right) = 2y D_{\lambda_i} \hat{\theta}\left(2 \int_0^{\lambda_i}\right) + \mathcal{O}(y^3), \quad (\text{B.1})$$

This is a simple zero if $D_{\lambda_i^-} \hat{\theta}(2 \int_0^{\lambda_i^-})$ is finite and at least a triple zero (since the function is odd) if it vanishes. The path $2 \int_0^{\lambda_i}$ defines a closed cycle and we may write $2 \int_0^{\lambda_i} \equiv m + \Pi n$ for $m, n \in \mathbb{Z}$. Using the theta function identities from the main text and recalling that $2 \int_0^\infty = l + \Pi k$, one may write

$$\hat{\theta}\left(z + 2 \int_0^{\lambda_i}\right) = C \exp\{\pi i (k^\top - 2n^\top) z\} \theta\left(z + \int_0^\infty\right), \quad (\text{B.2})$$

where C is a nonzero constant. Since $\theta(\int_0^\infty)$ vanishes, It is thus apparent that

$$\begin{aligned} D_{\lambda_i} \hat{\theta} \left(2 \int_0^{\lambda_i} \right) &= D_{\lambda_i} \hat{\theta} \left(z + 2 \int_0^{\lambda_i} \right) \Big|_{z=0} \\ &= C \exp \{ \pi i (l^\top - 2n^\top) z \} D_{\lambda_i} \theta \left(z + \int_0^\infty \right) \Big|_{z=0} = C D_{\lambda_i} \theta \left(\int_0^\infty \right). \end{aligned} \quad (\text{B.3})$$

In particular, $D_{\lambda_i} \theta(\int_0^\infty) = 0$ is equivalent to $D_{\lambda_i} \hat{\theta}(2 \int_0^{\lambda_i}) = 0$. It therefore suffices to show that the directional derivative, $D_{\lambda_i^-} \theta(\int_0^\infty)$, vanishes. To do so, we shall employ *Fay's trisecant identity*, see e.g., [15]. A corollary of the trisecant identity states that

$$D_\lambda \ln \frac{\theta(z)}{\theta(z + \int_\eta^\rho)} = -D_\lambda \ln \frac{\theta(a + \int_\eta^\lambda)}{\theta(a + \int_\rho^\lambda)} - \frac{D_\lambda \theta(a) \theta(a + \int_\rho^\eta)}{\theta(a + \int_\rho^\lambda) \theta(a + \int_\lambda^\eta)} \frac{\theta(z + \int_\eta^\lambda) \theta(z + \int_\lambda^\rho)}{\theta(z) \theta(z + \int_\eta^\rho)}, \quad (\text{B.4})$$

where a is a non-singular zero of theta. Let us choose $\rho = 0$, $a = \int_0^\infty$ and $z = \int_0^\infty + \int_0^\lambda$:

$$D_\lambda \ln \frac{\theta(\int_0^\infty + \int_0^\lambda)}{\theta(\int_0^\infty + \int_\eta^\lambda)} = -D_\lambda \ln \frac{\theta(\int_0^\infty + \int_\eta^\lambda)}{\theta(\int_0^\infty + \int_0^\lambda)} - \frac{D_\lambda \theta(\int_0^\infty) \theta(\int_0^\infty + \int_0^\eta)}{\theta(\int_0^\infty + \int_0^\lambda) \theta(\int_0^\infty + \int_\lambda^\eta)} \frac{\theta(\int_\eta^\infty + 2 \int_0^\lambda) \theta(\int_0^\infty)}{\theta(\int_0^\infty + \int_0^\lambda) \theta(\int_0^\infty + \int_\eta^\lambda)}. \quad (\text{B.5})$$

It is apparent that the left-hand side cancels with the first term of the right-hand side. Let us examine the analytic structure of the second term on the right-hand side. Taking λ and η to be regular points on the surface we have a simple zero due to the $\theta(\int_0^\infty)$ factor in the numerator. The identity is thus satisfied.

The situation is different when λ is an odd finite branch point, since $\theta(\int_0^\infty + \int_0^{\lambda_i})$ in the denominator vanishes. This may be seen by noting that $\theta(z) = 0$ if and only if $z \in W_{g-1} + \kappa$, where W_{g-1} is the image of integral divisors of degree $g-1$ on Σ under the Abel-Jacobi mapping and κ is the *vector of Riemann constants* [18]. The vector of Riemann constants may be given by the image of the divisor of the odd branch point characteristics under the Abel-Jacobi mapping, i.e., $\kappa = \int_0^\infty + \sum_j^{g-1} \int_0^{\lambda_j^-}$. We may write

$$\int_0^{\lambda_i} + \int_0^\infty = \int_0^{\lambda_i} - \sum_j^{g-1} \int_0^{\lambda_j^-} + \kappa. \quad (\text{B.6})$$

It is apparent that $\int_0^{\lambda_i} - \sum_j^{g-1} \int_0^{\lambda_j^-} \in W_{g-1}$ if and only if λ_i is an odd *finite* branch point.

Thus, in order for the identity (B.5) to be satisfied for $\lambda = \lambda_i^-$, we find

$$D_{\lambda_i^-} \theta \left(\int_0^\infty \right) = 0, \quad (\text{B.7})$$

as required, and indeed it is at least a double zero. We emphasize that the same procedure does not apply for $D_\infty \theta(\int_0^\infty)$, or indeed for the even branch points.

We therefore find that $\hat{\theta}(2 \int_0^\lambda)$ has a zero of order (at least) three at the odd finite branch points. As there are $g-1$ finite odd branch points and $g+1$ other branch points which are also zeros, we find a total of at least $3(g-1) + (g+3) = 4g$, zeros. But as stated at the end of section 3, this is exactly the number of zeros of $\hat{\theta}(2 \int_0^\lambda)$, so we have identified the correct order of them all.

C Equivalence of the sigma-model and Pohlmeyer reduced Lax operators

It is straightforward to see that the currents (4.7) are given in terms of the reduced currents (2.7) via

$$\mathcal{J}(x) = \frac{1}{1-x} (h^\dagger)^{-1} (j + \bar{j}^\dagger) h^\dagger, \quad \bar{\mathcal{J}}(x) = \frac{1}{1+x} (h^\dagger)^{-1} (\bar{j} + j^\dagger) h^\dagger. \quad (\text{C.1})$$

with h from (2.3). Taking the Lax connection M in (4.2) to be given by the generalized Maurer-Cartan forms (4.7) the holomorphic part of the Lax equation may be written as

$$\partial L = \frac{1}{1-x} [L, (h^\dagger)^{-1} (j + \bar{j}^\dagger) h^\dagger]. \quad (\text{C.2})$$

Let us define $\mathcal{L} \equiv h^\dagger L (h^\dagger)^{-1}$. Then (C.2) is equivalent to

$$\partial \mathcal{L} = \frac{1}{1-x} \left([\mathcal{L}, j] + x [\mathcal{L}, \bar{j}^\dagger] \right), \quad (\text{C.3})$$

which is none other than

$$\partial \mathcal{L} = [\mathcal{L}, J]. \quad (\text{C.4})$$

Therefore, if we solve for \mathcal{L} we may obtain L by simple gauge transformation and furthermore the algebraic curve of L and \mathcal{L} are identical.

D Factorized string method for genus-one

We have constructed the algebraic curve for all-genus. This was done for a Lax operator with respect to the Pohlmeyer connection. We now construct the Lax operator with respect to the sigma-model connection, for the case of genus-one. The resulting expression will be given in terms of the sigma-model spectral parameter x , rather than the Pohlmeyer spectral parameter $(1+x)/(1-x)$.

The factorized string method of [7] separates the functional dependence of the connections on the two world-sheet coordinates σ and τ (where $w = \sigma + i\tau$). This can be implemented for the genus-one solutions where as we will show, the Maurer-Cartan forms may be expressed as

$$\mathcal{J}_\sigma(\sigma, \tau) = S^{-1}(\sigma) \mathcal{J}_\sigma^0(\tau) S(\sigma), \quad \mathcal{J}_\tau(\sigma, \tau) = S^{-1}(\sigma) \mathcal{J}_\tau^0(\tau) S(\sigma). \quad (\text{D.1})$$

In the case of interest $S \in \text{SL}(2; \mathbb{C})/\text{SU}(2)$. As a consequence of this factorization, a Lax operator may be defined as $L \equiv \mathcal{J}_\sigma(x) - \mathcal{J}_\tau^L$. Here $\mathcal{J}_\sigma(x) \equiv \frac{1}{1-x^2} (\mathcal{J}_\sigma - ix \mathcal{J}_\tau)$ and $\mathcal{J}_\tau^L \equiv S^{-1} \partial_\tau S = \text{constant}$.⁵ Note that this Lax operator is with respect to the generalized Maurer-Cartan connection, cf., the discussion in section 4, i.e., it satisfies

$$\partial_\sigma L = [L, \mathcal{J}_\sigma(x)], \quad \partial_\tau L = [L, \mathcal{J}_\tau(x)]. \quad (\text{D.2})$$

⁵This is consistent with the definition of the generalized currents, (4.7), in terms of w and \bar{w} .

The first step is to write down the explicit expressions for the Maurer-Cartan forms for these solutions. It follows from the definitions of the currents and (4.8) that

$$\begin{aligned}\mathcal{J}_\sigma^\star &= \frac{\theta^2(\int_0^\lambda)}{D_\infty \hat{\theta}(0) \hat{\theta}(2 \int_0^\lambda) \hat{\theta}^2(z)} \begin{pmatrix} \lambda \theta_+ \theta_- - \hat{\theta}_+ \hat{\theta}_- & (\lambda \theta_-^2 + \hat{\theta}_-^2) e^{-2\tilde{\mu}\sigma - 2\tilde{\nu}\tau} \\ (-\lambda \theta_+^2 - \hat{\theta}_+^2) e^{2\tilde{\mu}\sigma + 2\tilde{\nu}\tau} & -\lambda \theta_+ \theta_- + \hat{\theta}_+ \hat{\theta}_- \end{pmatrix}, \\ \mathcal{J}_\tau^\star &= \frac{\theta^2(\int_0^\lambda)}{D_\infty \hat{\theta}(0) \hat{\theta}(2 \int_0^\lambda) \hat{\theta}^2(\zeta)} \begin{pmatrix} -\lambda \theta_+ \theta_- - \hat{\theta}_+ \hat{\theta}_- & (-\lambda \theta_-^2 + \hat{\theta}_-^2) e^{-2\tilde{\mu}\sigma - 2\tilde{\nu}\tau} \\ (\lambda \theta_+^2 - \hat{\theta}_+^2) e^{2\tilde{\mu}\sigma + 2\tilde{\nu}\tau} & \lambda \theta_+ \theta_- + \hat{\theta}_+ \hat{\theta}_- \end{pmatrix},\end{aligned}\quad (\text{D.3})$$

where we wrote the complex conjugates of the currents for presentational purposes and we defined $\tilde{\mu}(\lambda)\sigma + \tilde{\nu}(\lambda)\tau \equiv \mu(\lambda)z + \nu(\lambda)\bar{z}$,⁶ and we have introduced the shorthand, $\theta_\pm \equiv \theta(z \pm \int_0^\lambda)$ and $\hat{\theta}_\pm \equiv \hat{\theta}(z \pm \int_0^\lambda)$. We treat λ as a constant here.

We have thus far failed to mention anything about the form of $z(w, \bar{w})$. It is related to the world-sheet coordinates by [8]

$$z \equiv 2 [\omega(\infty)w + \omega(0)\bar{w}], \quad (\text{D.4})$$

which for the genus-one case, reduces to

$$z = 4i\omega(\infty)\tau. \quad (\text{D.5})$$

In particular, this ensures that none of the theta functions are a function of σ . This may be exploited to factorize the currents with respect to σ . Consider the matrix

$$S(\sigma) = \begin{pmatrix} e^{\tilde{\mu}\sigma} & 0 \\ 0 & e^{-\tilde{\mu}\sigma} \end{pmatrix} \in \text{SL}(2; \mathbb{C})/\text{SU}(2). \quad (\text{D.6})$$

It is apparent that this matrix satisfies (D.1), with

$$\mathcal{J}_\sigma^{0\star} = \frac{\theta^2(\int_0^\lambda)}{D_\infty \hat{\theta}(0) \hat{\theta}(2 \int_0^\lambda) \hat{\theta}^2(z)} \begin{pmatrix} \lambda \theta_+ \theta_- - \hat{\theta}_+ \hat{\theta}_- & (\lambda \theta_-^2 + \hat{\theta}_-^2) e^{-2\tilde{\nu}\tau} \\ (-\lambda \theta_+^2 - \hat{\theta}_+^2) e^{2\tilde{\nu}\tau} & -\lambda \theta_+ \theta_- + \hat{\theta}_+ \hat{\theta}_- \end{pmatrix}, \quad (\text{D.7})$$

$$\mathcal{J}_\tau^{0\star} = \frac{\theta^2(\int_0^\lambda)}{D_\infty \hat{\theta}(0) \hat{\theta}(2 \int_0^\lambda) \hat{\theta}^2(\zeta)} \begin{pmatrix} -\lambda \theta_+ \theta_- - \hat{\theta}_+ \hat{\theta}_- & (-\lambda \theta_-^2 + \hat{\theta}_-^2) e^{-2\tilde{\nu}\tau} \\ (\lambda \theta_+^2 - \hat{\theta}_+^2) e^{2\tilde{\nu}\tau} & \lambda \theta_+ \theta_- + \hat{\theta}_+ \hat{\theta}_- \end{pmatrix}. \quad (\text{D.8})$$

Additionally

$$\mathcal{J}_S^L = \tilde{\mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \text{constant}. \quad (\text{D.9})$$

As L is traceless, the spectral curve may be given by

$$y^2 = \frac{1}{2} \text{tr} L^2, \quad (\text{D.10})$$

⁶Again, the specific forms of $\tilde{\mu}$ and $\tilde{\nu}$ are not important, only that they are functions of the spectral parameter.

or more explicitly

$$y^2 = \frac{1}{2(1-x^2)^2} \text{tr} \left[(\mathcal{J}_\sigma - \mathcal{J}_S^L)^2 + x \{2i(\mathcal{J}_S^L - \mathcal{J}_\sigma)\mathcal{J}_\tau\} + x^2 \{2(\mathcal{J}_\sigma - \mathcal{J}_S^L)\mathcal{J}_S^L - \mathcal{J}_\tau^2\} \right. \\ \left. + x^3 \{-2i\mathcal{J}_S^L\mathcal{J}_\tau\} + x^4 \{(\mathcal{J}_S^L)^2\} \right]. \quad (\text{D.11})$$

The calculation of the coefficients is straightforward following section 5. The result is

$$y^2 = \frac{1}{(1-x^2)^2} \left\{ 2 \left(\lambda - \frac{1}{\lambda} \right) - 2 \left(a - \frac{1}{a} \right) + x \left[-4 \left(\lambda - \frac{1}{\lambda} \right) \right] + x^2 \left[4 \left(a - \frac{1}{a} \right) \right] \right. \\ \left. + x^3 \left[4 \left(\lambda - \frac{1}{\lambda} \right) \right] + x^4 \left[-2 \left(\lambda - \frac{1}{\lambda} \right) - 2 \left(a - \frac{1}{a} \right) \right] \right\}, \quad (\text{D.12})$$

which is birationally equivalent to

$$y^2 = \lambda \frac{1+x}{1-x} - \frac{1}{\lambda} \frac{1-x}{1+x} - a + \frac{1}{a}. \quad (\text{D.13})$$

We see that solutions with different values of λ lead to birationally equivalent curves. Let us absorb $\tilde{\lambda} \equiv \lambda \frac{1+x}{1-x}$ and $\tilde{y} \equiv y/\tilde{\lambda}$ which brings the curve to the standard form

$$\tilde{y}^2 = \tilde{\lambda} \left(\tilde{\lambda} - a \right) \left(\tilde{\lambda} + \frac{1}{a} \right). \quad (\text{D.14})$$

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